

GENERALIZED KP HIERARCHY FOR SEVERAL VARIABLES

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ABSTRACT. Following the techniques of M. Sato (see [Sa]), a generalization of the KP hierarchy for more than one variable is proposed. An approach to the classification of solutions and a method to construct algebraic solutions is also offered.

1. INTRODUCTION

After the remarkable papers of Krichever ([Kr]), Mulase ([Ml]), Segal-Wilson ([SW]) and Sato (see [Sa] and references therein) the KP hierarchy was extensively studied and a lot of important results were eventually given in many related problems. Let us cite some of their topics: infinitesimal transformations of soliton equations ([DJKM]); characterization of Jacobian varieties ([Sh]); relation of commutative rings of differential operators and abelian varieties ([N]); and more recently, study of the moduli space of pointed curves ([MP]); algebraic solutions of the multicomponent KP hierarchy ([P11]); generalization of the Krichever correspondence for varieties of dimension greater than 1 ([O]); generalization of the KP formalism for pseudodifferential operators in several variables ([Pa]); etc. .

This paper aims at generalizing the theory of the KP hierarchy for several variables following Sato's techniques (see, for instance, [Sa]). We think that our approach will be useful to study the above cited topics in greater generality.

The two main parts in Sato's approach to the KP hierarchy are the use of a universal grassmann manifold as classifying space for the solutions, on the one hand, and the construction of solutions from some algebro-geometric data through the Krichever map, on the other. Usually, these two points have been studied separately. We offer a generalization which keeps both aspects together as it was proposed in

1991 Mathematics Subject Classification: 35Q53, 58F07.

Keywords: KP hierarchy, Sato Grassmannian.

This work is partially supported by the CICYT research contract n. PB96-1305 and Castilla y León regional government contract SA27/98.

§3.4 of [Sa] for dimension greater than 1; in this sense, it is closer to the methods employed in [DJKM, Pa, O, Pl1].

Let us summarize the contents of the paper. The second section begins with some facts on pseudodifferential operators and the definition of KP hierarchy for N variables (see, for instance, [Pa]). Introducing the notion of wave function for this hierarchy, one proves that this hierarchy can be written as the compatibility condition of a system of differential equations (Theorem 2.4).

In the third section, we study the set of wave functions in terms of an infinite dimensional Grassmannian, which is a generalization of those given in [AMP, Pl1, Sa, SW]. Theorems 3.5 and 3.6 generalize Sato's classification of solutions of the KP hierarchy: there is a 1-1 correspondence between the set of wave functions and the points of the infinite Grassmannian.

The last section offers a method for constructing solutions for the new hierarchy starting with some algebro-geometric data (Theorem 4.2). Similarly to the standard KP, the solutions obtained in this way are finite gap. This construction generalizes the Krichever map. It would be interesting to include Osipov's generalization ([O]) in our framework.

I would like to express my gratitude to Prof. G. B. Segal for inviting me to the DPMMS at University of Cambridge (UK) where most of this work has been done.

2. THE NEW HIERARCHY

2.A. Pseudodifferential Operators. Let us begin recalling some standard definitions and properties of pseudo-differential operators (pdo) for N variables, $N \geq 1$ (see [Pa]).

Multiindexes will be denoted with greek letters and the following notation will be used:

- the entries of a multiindex α will be denoted with subindexes, i.e. $(\alpha_1, \dots, \alpha_N)$;
- 0 denotes the multiindex $(0, \dots, 0)$;
- for α and β we say that $\alpha \subseteq \beta$ if $\alpha_i \leq \beta_i$ for all i ;
- $\alpha \subset \beta$ means that $\alpha \subseteq \beta$ and that $\alpha \neq \beta$;
- \leq will denote the reverse lexicographic order in \mathbb{Z}^N ; that is, $\alpha \leq \beta$ if $\sum_{i=1}^N \alpha_i k^i \leq \sum_{i=1}^N \beta_i k^i$ for all $k \gg 0$.

For unknowns x_1, \dots, x_N , let $\mathbb{C}_x := \mathbb{C}((x_1)) \dots ((x_N))$ be the N -dimensional local field of iterated Laurent series; that is, $\mathbb{C}((x_1)) \dots ((x_i))$ is defined to be the quotient field of $\mathbb{C}((x_1)) \dots ((x_{i-1}))[[x_i]]$. Let $\mathbb{C}[[x]]$ denote $\mathbb{C}[[x_1, \dots, x_N]]$.

Now, one considers the $\mathbb{C}[[x]]$ -module of pdo's:

$$\mathcal{P} := \left\{ \sum a_\alpha \partial^\alpha \mid a_\alpha \in A, n \in \mathbb{Z} \right\} \subseteq \mathbb{C}[[x]]((\partial_1^{-1})) \dots ((\partial_N^{-1}))$$

where $\partial_i := \frac{\partial}{\partial x_i}$ ($1 \leq i \leq N$) and $\partial^\alpha := \partial_1^{\alpha_1} \dots \partial_N^{\alpha_N}$.

Since $[\partial_i, \partial_j] = 0$, the following generalization of the Leibnitz rule:

$$\left(\sum_\alpha a_\alpha \partial^\alpha \right) \left(\sum_\beta b_\beta \partial^\beta \right) := \sum_{\alpha, \beta} \sum_{0 \subseteq \gamma} \prod_{i=1}^N \binom{\alpha_i}{\gamma_i} a_\alpha (\partial^\gamma b_\beta) \partial^{\alpha+\beta-\gamma}$$

endows \mathcal{P} with a \mathbb{C} -algebra structure. Mapping a pdo $P = \sum_\alpha a_\alpha \partial^\alpha$ to $P_+ := \sum_{\alpha \geq 0} a_\alpha \partial^\alpha$ one obtains an endomorphism $\mathcal{P} \rightarrow \mathcal{P}$ whose image (resp. kernel) will be denoted by \mathcal{P}_+ (resp. \mathcal{P}_-).

2.B. KP for N variables. Let $\mathbb{C}[[t]]$ denote $\mathbb{C}[[\{t_\alpha\}_{0 \subset \alpha}]]$ and ∂_α be $\frac{\partial}{\partial t_\alpha}$. For pdo L_1, \dots, L_N let L^α denote $L_1^{\alpha_1} \dots L_N^{\alpha_N}$.

Definition 2.1. *The KP hierarchy for N variables, $KP(N)$, is the following Lax system:*

$$\begin{cases} \partial_\alpha L_i = [(L^\alpha)_+, L_i] \\ [L_i, L_j] = 0 \end{cases} \quad 1 \leq i, j \leq N, 0 \subset \alpha \quad (2.2)$$

where:

$$L_i = \partial_i + \sum_{\alpha \subset 0} u_{i\alpha}(t) \partial^\alpha \in \mathcal{P} \otimes \mathbb{C}[[t]] \quad i = 1, \dots, N \quad (2.3)$$

A formal oscillating function, $w(t, x)$, over this ring is a formal expression of the following type:

$$\left(1 + \sum_{0 \subset \alpha} a_\alpha(t) x^\alpha \right) \cdot e^{\xi(t, x)}$$

where $e^{\xi(t, x)} := \exp(\sum_{0 \subset \beta} t_\beta x^{-\beta})$, $t = \{t_\beta \mid 0 \subset \beta\}$ are “time” variables and $a_\alpha(t) \in \mathbb{C}[[\{t_\alpha\}_{0 \subset \alpha}]]$.

The KP hierarchy as defined in 2.2 consists of a set of deformation equations for the operators L_i . Similarly to §1.2 of [Sh], it admits other equivalent formulations.

Theorem 2.4. *The following conditions are equivalent:*

1. $\{L_1, \dots, L_N\}$ are pdo of the type 2.3 and satisfy 2.2;
2. there exists a pdo $S \in 1 + \mathcal{P}_- \otimes \mathbb{C}[[t]]$ such that:

$$\begin{cases} L_i S = S \partial_i \\ \partial_\alpha S = -(S \partial^\alpha S^{-1})_- S \end{cases} \quad \begin{matrix} 1 \leq i \leq N \\ 0 \subset \alpha \end{matrix} \quad (2.5)$$

3. $\{L_1, \dots, L_N\}$ admits a wave function; that is, a formal oscillating function $w(t, x)$ such that:

$$\begin{cases} L_i w = x_i \cdot w & i = 1, \dots, N \\ \partial_\alpha w = (L^\alpha)_+ w & 0 \subset \alpha \end{cases} \quad (2.6)$$

Proof. The equivalence of 2 and 3 is formal since $w(t, x)$ and $S(t, \partial)$ are related through $w(t, x) = S(t, x)e^{\xi(t, x)}$.

From [Pa] one knows that 2 implies 1. Conversely, if $\{L_1, \dots, L_N\}$ are of the form 2.3 and commute pairwise, then there exists $S \in 1 + \mathcal{P}_-$ such that $L_i = S\partial_i S^{-1}$ ([Pa]), and such S is unique up to right multiplication by a pdo in $1 + \mathcal{P}_-$ with constant coefficients. Further, arguments similar to those of §1.2 of [Sh] show that equation 2.5 determines recursively the coefficients of S . \square

Definition 2.7. A finite gap wave function for the KP(N) hierarchy is a wave function $w(t, x)$ such that $\partial_\alpha w = 0$ for a finite number of α 's.

3. CLASSIFICATION OF SOLUTIONS

From now on V will denote the complex vector space \mathbb{C}_x endowed with the filtration $V_n := x_N^n \mathbb{C}((x_1)) \dots ((x_{N-1}))[[x_N]]$ with $n \in \mathbb{Z}$.

For a subspace A of V , one observes that:

$$\begin{aligned} d_A : \mathbb{Z} &\rightarrow \mathbb{Z} \cup \{\infty\} \\ n &\mapsto \dim(V_n/A \cap V_n) \end{aligned}$$

is a decreasing function.

Definition 3.1. We consider the linear topology in V given by the basis, \mathcal{B} , of neighbourhoods of (0): the set of proper subspaces $A \subset V$ such that $d_A(n)$ is finite for all n and converges to zero.

Given $A \in \mathcal{B}$ and a proper subspace of V , B , the following properties hold:

- if $A \subseteq B$, then $B \in \mathcal{B}$;
- if $B \subseteq A$ is of finite codimension, then $B \in \mathcal{B}$;
- if $B \in \mathcal{B}$, then $A \cap B \in \mathcal{B}$.

Definition 3.2. The Grassmannian of the pair (V, \mathcal{B}) is the \mathbb{C} -scheme representing the functor:

$$S \rightsquigarrow \left\{ \begin{array}{l} \text{submodules } U \subset \hat{V}_S \text{ such that } U \oplus \hat{A}_S \rightarrow \hat{V}_S \\ \text{is an isomorphism for a subspace } A \in \mathcal{B} \end{array} \right\}$$

(where \hat{A}_S denotes the completion of $A \otimes_{\mathbb{C}} \mathcal{O}_S$).

Recall that in [AMP, MP, Pl2] the Grassmannian of $(V = \mathbb{C}((x)), V_n = x^n \mathbb{C}[[x]])$ is defined using the notion of subspace commensurable with V_0 and observe that such notion is more restrictive. However, the existence of the scheme $\text{Gr}(V)$ is, in both cases, a direct consequence of the following:

Lemma 3.3. *Let A be an element of \mathcal{B} . Then, \hat{A} is an inverse limit of finite dimensional \mathbb{C} -vector spaces.*

Proof. Note that \mathcal{B} gives a basis of the induced topology on A ; namely $\{A \cap B \mid B \in \mathcal{B}\}$.

For a given subspace $B \in \mathcal{B}$, the sets:

$$\begin{aligned}\mathcal{B}_A &:= \{C \subseteq A \text{ s.t. } \dim A/C < \infty\} \\ \mathcal{B}_{A,B} &:= \{C \in \mathcal{B}_A \text{ s.t. } A \cap B \subseteq C \subseteq A\}\end{aligned}$$

satisfy $\mathcal{B}_{A,B} \subseteq \mathcal{B}_A \subseteq \mathcal{B}$. The commutativity of the following diagram:

$$\begin{array}{ccc}\hat{A} = \varprojlim_{B \in \mathcal{B}} A/(A \cap B) & \xrightarrow{\quad} & \varprojlim_{C \in \mathcal{B}_A} A/(A \cap C) \\ \downarrow & & \downarrow \\ A/(A \cap B) & \hookrightarrow & \varprojlim_{C \in \mathcal{B}_{A,B}} A/(A \cap C)\end{array}$$

implies that the horizontal arrow on the top is an isomorphism, and the conclusion follows. \square

Motivated by the properties of Baker-Ahkizer functions for the standard KP hierarchy (see [Kr, MP, SW]), we give the following:

Definition 3.4. *Given a point $U \in \text{Gr}(V)$ and a basis $\{u_i\}_{i \in I}$ of it, define the Baker-Ahkizer function, ω_U , as the formal sum:*

$$\omega_U(t, x) := \sum_{i \in I} t_{v(u_i)} u_i$$

where $v : V \rightarrow \mathbb{Z}^N$ maps $\sum_{\alpha} a_{\alpha} x^{\alpha}$ to the multiindex α such that $a_{\beta} = 0$ for all $\beta < \alpha$.

Define U_0 to be the subspace $\mathbb{C}[x_1^{-1}, \dots, x_N^{-1}]$ and consider $A_0 \subset V$ satisfying: $U_0 \oplus A_0 \simeq V$; $x^{\alpha} \in A_0$ for all multiindex $\alpha \not\leq 0$; and $A_0 \in \mathcal{B}$. Let F_0 be the open subscheme $\text{Hom}(U_0, \hat{A}_0) \subset \text{Gr}(V)$.

Theorem 3.5. *There is a natural map from the set of wave functions for the KP(N) hierarchy to $F_0 \subset \text{Gr}(V)$.*

Proof. Let $w(t, x) = S(t, \partial_t)e^{\xi(t, x)}$ be a formal oscillating function with $S(t, x) = 1 + \sum_{\beta < 0} a_\beta x^\beta$.

If w is a wave function for the KP(N) hierarchy and $0 \subseteq \alpha$ is a multiindex, one has:

$$\partial_\alpha w = x^{-\alpha} w - (L^\alpha)_- w = (x^{-\alpha} S - (S \partial^\alpha S^{-1})_- S) e^{\xi(t, x)}$$

and therefore:

$$(\partial_\alpha w)|_{t=0} = x^{-\alpha} + (\text{higher order terms})$$

This means that $(\partial_\alpha w)|_{t=0}$ generates a point of $F_0 \subset \text{Gr}(V)$ as α varies. \square

Remark 1. This map is equivariant w.r.t. the action by right multiplication of $1 + \mathbb{C}((\partial_1^{-1})) \dots ((\partial_N^{-1}))_-$ on the set of wave functions and that of $1 + \mathbb{C}((x_1^{-1})) \dots ((x_N^{-1}))_-$ on $\text{Gr}(V)$.

Theorem 3.6. *If U is a point of F_0 , then its Baker-Akhiezer function is a wave function for the KP(N).*

Proof. Let K be the field $\mathbb{C}((x_1)) \dots ((x_{N-1}))$. If U is a subspace in F_0 , then the K -vector space $\tilde{U} \subset K((x_N))$ given by the image of $K \otimes_{\mathbb{C}} U \rightarrow K((x_N))$ is a point of $\text{Gr}(K((x_N)), K[[x_N]])$ (see [AMP, MP] for the definition of the Grassmannian). Since $K((x_N))$ carries a non-degenerate bilinear pairing, $<, >$, induced by the residue at $x_N = 0$, we consider $(\tilde{U})^\perp \in \text{Gr}(K((x_N)), K[[x_N]])$. Let $\psi_U(s, x_N)$ be the wave function of $(\tilde{U})^\perp$.

From the definition of $\omega_U(t, x)$ and from the properties of $\psi_U(s, x_N)$, one obtains that:

$$< \omega_U(t, x), \psi_U(s, x_N) > = 0 \quad \forall t, s$$

This identity implies that (see [DJKM]):

$$\left((\partial_\alpha + (L^\alpha)_-) S(t, \partial_t) R^*(t, \partial_N) \right)_- = 0 \quad \forall \alpha$$

where $\omega_U(t, x) = S(t, \partial_t)e^{\xi(t, x)}$, $\psi_U(s, x_N) = R(s, \partial_s)e^{\sum_{i>0} s_i x_N^{-i}}$, and R^* is the adjoint operator of R . Then, it follows that:

$$(\partial_\alpha + (L^\alpha)_-) S(t, \partial_t) = 0 \quad \forall \alpha$$

\square

Remark 2. Once one has related the KP(N) hierarchy with the infinite Grassmannian $\text{Gr}(V)$, one observes that the groups acting on it will give symmetries of the hierarchy (see [DJKM] for the case of the KP hierarchy).

Further, there is a natural 2-cocycle given by $\det(\delta_B^{-1} \circ \delta_A) \in H^0(F_A \cap F_B, \mathcal{O}_{F_A \cap F_B}^*)$, where \mathcal{L} is the universal submodule and δ_A is the morphism $\mathcal{L} \oplus A \rightarrow V$. Then, for a group acting on $\text{Gr}(V)$ the line bundle associated to this cocycle defines central extensions of the group and of its Lie algebra.

The case of the Lie algebra of the group $\text{Aut}_{\mathbb{C}\text{-alg}} \hat{V}$ has beautiful properties when $N = 1$ due to its relation with the Virasoro algebra (see [MP2]). However, the $N > 1$ situation differs substantially since the Lie algebra has no non-trivial central extensions. This fact follows from [RSS], since that Lie algebra of $\text{Aut}_{\mathbb{C}\text{-alg}} \hat{V}$ has a dense subalgebra with generators $\{L_\alpha^a \mid \alpha \in \mathbb{Z}^N, 1 \leq a \leq N\}$ and the following Lie bracket $[L_\alpha^a, L_\beta^b] = \beta_a L_{\alpha+\beta}^b - \alpha_b L_{\alpha+\beta}^a$.

4. FINITE GAP SOLUTIONS

From [Kr, SW] it is known that for the standard KP hierarchy the Krichever morphism provides a way to construct solutions starting with the geometric data (C, p, α) where C is an algebraic integral curve over \mathbb{C} , $p \in C$ is a smooth point, and α is an isomorphism $\hat{\mathcal{O}}_{C,p} \simeq \mathbb{C}[[x_0]]$.

Now, we will see how finite gap solutions for the KP(N) hierarchy might be constructed from some algebro-geometric data.

Definition 4.1. *An algebro-geometric datum for the KP(N) consists of $(X, p, \alpha, \{Y_1, \dots, Y_N\})$ where X is a N -dimensional integral regular projective scheme, p is a point, α is an isomorphism $\hat{\mathcal{O}}_{X,p} \simeq \mathbb{C}[[x_1, \dots, x_N]]$, and $\{Y_1, \dots, Y_N\}$ is a ordered set of Weil divisors such that: $p \in \cap_i Y_i$, and $x_i = 0$ is the local equation of Y_i in a neighbourhood of p .*

Observe that given an algebro-geometric datum $(X, p, \alpha, \{Y_1, \dots, Y_N\})$, the morphism $\mathcal{O}_{X,p} \hookrightarrow \hat{\mathcal{O}}_{X,p} \xrightarrow{\sim} \mathbb{C}[[x_1, \dots, x_N]]$ induces a map from the function field of X , Σ , to V . Moreover, given a function $f \in \Sigma$, one can define $v(f) \in \mathbb{Z}^N$ as the smallest (w.r.t. the order \leq) exponent occurring in the image of f by the natural map $\Sigma \rightarrow V$.

Now, the map $\Sigma \hookrightarrow V$ and Serre's vanishing theorem allow an easy generalization of Krichever's construction ([Kr] in the following form:

Theorem 4.2. *Let $\mathfrak{X} = (X, p, \alpha, \{Y_1, \dots, Y_N\})$ be an algebro-geometric datum for the KP(N). Then, the \mathbb{C} -vector space:*

$$A_{\mathfrak{X}} := \{f \in \bigcup_{0 \subseteq \alpha} H^0(X, \mathcal{O}_X(\sum_i \alpha_i Y_i) \mid 0 \subseteq v(f)\}$$

is a point of $\text{Gr}(V)$ and its wave function is a finite gap solution of the KP(N) hierarchy.

To finish this section let us point out two remarks. First, it could be interesting to adapt Osipov's generalization of the Krichever correspondence ([O]) to this kind of algebro-geometric data. Second, one wonders about the possible generalizations of the Burchnell-Chaundy theory for studying rings of commuting differential operators within this framework.

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